國立新竹教育大學 100 學年度碩、博士班招生考試試題

所別：應用數學系碩士班

科目：線性代數(本科總分：150 分)

※ 請橫書作答

1. Given the matrix $A=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$

   (a) Find the determinant of A.(6 points)

   (b) Find the eigenvalues and the corresponding eigenvectors.(10 points)

   (c) Find a matrix B such that $B^{-1}AB=D$ where D is diagonal matrix and write down the matrix D.(8 points)

   (d) Find A^{20}=? (6 points)

2. Let V=span(S) with the inner product $<f,g>=\int_0^{\pi} f(t)g(t)dt$ and $S=\{\sin t,\cos t,1,t\}$.

   (a) Apply the Gram-Schmidt process to the given S of the inner product V. Find an orthonormal basis β for V.(10 points)

   (b) If h(t)=2t+1, then compute the Fourier coefficients of the given vector relative to β.

   (10 points)

3. Let A and B be $n\times n$ matrices that are unitarily equivalent.

   (a) Prove that tr(A^*A)=tr(B^*B).(8 points)

   (b) Prove that

   1. $\sum_{i,j=1}^{n}|A_{ij}|^2=\sum_{i,j=1}^{n}|B_{ij}|^2$.(10 points)

   (c) Show that the matrices

   $\begin{bmatrix}1&2\\2&i\end{bmatrix}$ and $\begin{bmatrix}i&4\\1&1\end{bmatrix}$ are not unitarily equivalent.(7 points)

4. Consider the following system of linear equation,

$$\begin{cases}2x_1+3x_2+2x_3-x_4=0\\3x_1+2x_2+3x_3-2x_4=0\\-x_2+x_4=0\end{cases}.$$

   (a) Find the solution set W of the system of linear equation. (10 points.)

(b) Prove that W is a subspace for $\mathbb{R}^4$. What is the dimension of W? (10 points.)

(c) Find the solution set of the following nonhomogeneous system. (6 points.)

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 - x_4 = 6 \\ 3x_1 + 2x_2 + 3x_3 - 2x_4 = 6 \\ \quad - x_2 \qquad + x_4 = 0 \end{cases}.$$

5. Let $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be a function which is defined by

$$T(A) = AB^2 - BA$$

where $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be an ordered basis for $M_{2\times2}(\mathbb{R})$.

(a) Prove that T is a linear transformation from $M_{2\times2}(\mathbb{R})$ to $M_{2\times2}(\mathbb{R})$. (8 points.)

(b) Find bases for the **null space** $\mathrm{N}(T)$ and the **range** $\mathrm{R}(T)$ of T. (10 points.)

(c) Compute the **nullity** and **rank** of T. (6 points.)

(d) Compute $[T]_\alpha$ which is the matrix representation of T in the ordered basis α. (6 points.)

(e) Compute the determinant $\det([T]_\alpha)$. (6 points.)

6. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$.

(a) Prove that if S_1 is linearly dependent, then S_2 is linearly dependent. (8 points.)

(b) Consider the set

$$S = \{(1,0,0,-1), (5,0,3,-8), (0,0,1,-1), (5,3,-8,0)\}$$

in $\mathbb{R}^4$. Prove that S is linearly dependent. (5 points.)