國立新竹教育大學101學年度碩、博士班招生考試試題

所別：應用數學系碩士班

科目：線性代數(本科總分：150分)

※ 請橫書作答

1. Let A be an $n \times n$ matrix which is defined by

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \cdots & n \end{pmatrix}.$$

   (a) Find the **rank** and **nullity** of A. (10 points.)
   (b) Find the determinant of A. (8 points.)
   (c) Find the **trace** of A. (7 points.)
   (d) Find the characteristic polynomial of A. (10 points.)

2. In $P_2(\mathbb{R})$, consider the functions $f_1(x) = 1,\ f_2(x) = 1 + x,\ f_3(x) = 1 + x + x^2$.
   (a) Show that $\{f_1(x),\ f_2(x),\ f_3(x)\}$ form a basis for $P_2(\mathbb{R})$. (10 points.)
   (b) Find the unique representation for the function $f(x) = 6 + 3x + x^2$ in $P_2(\mathbb{R})$ as a linear combination of $f_1(x),\ f_2(x)$ and $f_3(x)$. (10 points.)

3. Consider the linear transformation $T:\ M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ which is defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+d) + bx + cx^2.$$

   Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$.
   (a) Compute $[T]_\beta^\gamma$. (10 points.)
   (b) Is T one-to-one? Justify your answer. (10 points.)

4. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + (-1)^n t^n.$$

   (a) Prove that $\det(A) \neq 0$ if and only if $a_0 \neq 0$. (7 points.)
   (b) Prove that if $\det(A) \neq 0$, then

$$A^{-1} = \frac{-1}{a_0}\left[a_1 I_n + \cdots + a_{n-1} A^{n-2} + (-1)^n A^{n-1} \right].\ (8\text{ points.})$$

(c) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 3 \end{pmatrix}$. Use (b) to compute A^{-1}. (10 points.)

5. Let $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}$.

   (a) Find the matrices P and Q such that PAQ is a diagonal matrix. (15 points.)

   (b) Find the trace of A^{10}. (7 points.)

   (c) Find the determinant of A^{10}. (8 points.)

   (d) Find the eigenvalues of $I + A + A^2 + A^3$. (10 points.)

6. Let T be a linear operator on $M_{2\times 2}(\mathbb{R})$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

   Determine whether T is normal, self-adjoint, or neither. (10 points.)