國立新竹教育大學九十九學年度研究所碩士班招生考試試題

所別：應用數學系碩士班

科目：線性代數（本科總分：150 分）

※請橫書作答

1. Let $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be linear and $[T]_\alpha = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ where $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

   Suppose $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, find $[T(A)]_\alpha$ and $T(A)$. (10 points)

2. Suppose $W = \left\{ \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 3 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 0 & 9 \end{bmatrix} \right\}$ and $U = \operatorname{span} W$. Find a subset of W that is a basis for U and explain. (15 points)

3. Let V, W and Z be finite-dimensional vector spaces with $\dim(V) = \dim(W)$ and $T: V \to W$, $U: W \to Z$ be linear.

   (i) Prove that the following conditions are equivalent (a) null space $N(T) = \{0\}$ (b) T is one-to-one (c) T is onto. (15 points)

   (ii) T is an isomorphism if and only if $T(\beta)$ is a basis of W for every basis β of V. (10 points)

   (iii) $rank(UT) \le rank(T)$. (10 points)

4. Let $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 1 & -4 & 4 & -6 \end{bmatrix}$. Does A^{-1} exist ? If it exists, try to find it. (15 points)

5. Let $A = \begin{bmatrix} 7/4 & -9/4 & -15/4 \\ 3/4 & 7/4 & 3/4 \\ 3/4 & -9/4 & -11/4 \end{bmatrix}$, $\lim\limits_{m\to\infty} A^m = ?$ (25 points)

6. Let $C = B - A$ be an $n\times n$ positive definite matrix. Prove that $|B| > |A|$. (20 points)

7. Prove or disprove.

   (i) Let A be an $n\times n$ real symmetric matrix, then $tr(A^T A) = \sum_{i=1}^{n} |\lambda_i|^2$, where λ_i's are the eigenvalues of A.

   (ii) Let A and B be $n\times n$ real matrices, x be an $n\times 1$ real matrix, then

$x^T Ax + x^T Bx = x^T (A+B)x$.

(iii) Let A and B be $n \times n$ real matrices, if $rank(A) = rank(B)$, then $rank(A^2) = rank(B^2)$. (30 points)