國立新竹教育大學102學年度碩、博士班招生考試試題

所別：應用數學系碩士班

科目：微積分(本科總分150分，含初等微積分、高等微積分)

本試題共2頁

※請橫書作答

1.(i) Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function. Assume that $c\in\mathbb{R}$ satisfies $f''(c)\neq 0$. Show that there exist $a,b\in\mathbb{R}$ and $a<c<b$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$. (10 points)

(ii)Please give an example to show that the assumption “ $f''(c)\neq 0$ ” in (i) is necessary. (5 points)

2.(i) Show that $\int_0^{\frac{\pi}{2}}\cos^{n+2}x\,dx=\frac{n+1}{n+2}\int_0^{\frac{\pi}{2}}\cos^n x\,dx$. (5 points)

(ii)Use (i) to calculate $\int_0^{\frac{\pi}{2}}\cos^n x\,dx$ for $n\in\mathbb{N}$. (5 points)

(iii)Use (ii) and the fact that $|\cos x|\leq 1$ to show

$\lim_{n\to\infty}\frac{1}{n}\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)^2=\pi$. (5 points)

3.Verifyvalues of the following integrals (20 points)

(i) $\int_2^3\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$ (ii) $\int_0^2\sqrt{\frac{x}{1+x^3}}dx$

(iii) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{1+2^{\cos x}}dx$ (iv) $\int_0^{\frac{\pi}{2}}\frac{3\sin x-4\cos x}{\sin x+2\cos x}dx$

4. Calculate the following limits (10 points)

(i) $\lim_{x\to 0}\left(\frac{x^2+x+1}{x^2+1}\right)^{2x+\frac{1}{x}}$ (ii) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

5.Verify the volume of the solid formed by revolving the region bounded by the graphs of $y=x^2-2x+2$, $y=0$, $x=-1$, and $x=2$ about the y-axis. (15 points)

6. Find a bounded continuous function $f:(0,1]\to\mathbf{R}$ such that f is not uniformly continuous and justify your answer. (15 points)

7. Suppose $a,b\in\mathbf{R}$, $a<b$ and $f_n:(a,b)\to\mathbf{R}$. Prove that if $\sum_{n=1}^{\infty}(-1)^n c_n$ converges and $\{f_n\}$ converges uniformly then $g_n(x):=c_n+x^3 f_n(x)$ converges uniformly on (a,b). (10 points)

8. Find an example of Riemann integrable function $f:[0,1]\to\mathbf{R}$ such that f has infinitely many discontinuous points and prove it.(15 points)

9. Suppose $a_n\ge 0$ for all $n\in\mathbf{N}$ and $\{x_n\}\subset\mathbf{R}$. Prove that

   (a) If $\sum_{n=1}^{\infty}a_n$ converges and $|x_n-x_{n-1}|<a_n$ for all $n\in\mathbf{N}$ then $\{x_n\}$ is a convergent sequence.(10 points)

   (b) Find an example of $\{a_n\}$ and $\{x_n\}$ with $\lim_{n\to\infty}a_n=0$ and $|x_n-x_{n-1}|<a_n$ but $\{x_n\}$ diverges to ∞.(10 points)

10. Suppose $\{f_n\}$ is a sequence of differentiable functions on $(0,2)$. Prove that if $\{f_n'\}$ converges uniformly on $(0,2)$ and $\lim_{n\to\infty}f_n(1)=0$ then $\{f_n\}$ converges uniformly on $(0,2)$.(15 points)